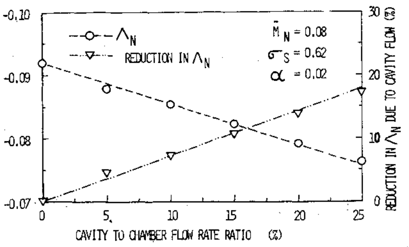

Fig. 2 Effect of cavity flow on the decay coefficient of submerged nozzle configuration of Ref. 3.

upon the nozzle decay coefficient Λ_N. However, for a given cavity depth, it can be observed that the nozzle decay coefficient Λ_N depends upon the cavity Mach number $\bar{M}_S$, and hence upon the cavity flow rate. To illustrate this dependence, the magnitude of Λ_N for the submerged nozzle configuration studied in Ref. 3 has been computed and the data are presented in Fig. 2. Also presented in this figure is the percent reduction in the value of Λ_N with increase in cavity flow when compared with no cavity flow in the test configuration. An examination of this figure indicates that, for example, when the cavity-to-chamber flow rate ratio is increased from 0 to 15%, a 10% reduction in Λ_N is observed. Suppose a given rocket motor, when using the nozzle under consideration without submergence was found to be marginally stable, then submerging this nozzle into the rocket combustor would undoubtedly result in a transition from stability to instability, due to the reduction in the effective nozzle damping.

The main conclusion that follows from this study on submerged nozzles is that the presence of the "reverse" cavity flow reduces the effective nozzle damping. This reduction in the damping capability of the nozzle offers one possible explanation to the observed[2] decrease in the stability of an experimental solid rocket motor when the nozzle was submerged into the combustor.

References

[1] Coates, R.L. and Horton, M.D., "Design Considerations for Combustion Instability," *Journal of Spacecraft and Rockets,* Vol. 6, March 1969, pp. 296-302.

[2] Micheli, P., private communication.

[3] Janardan, B.A., Daniel, B.R., and Zinn, B.T., "Damping of Axial Instabilities by Small-Scale Nozzles under Cold-Flow Conditions," *Journal of Spacecraft and Rockets,* Vol. 11, Dec. 1974, pp. 812-820.

[4] Cantrell, R.A. and Hart, R.W., "Interaction Between Sound and Flow in Acoustic Cavities; Mass, Momentum and Energy Considerations," *The Journal of the Acoustic Society of America,* Vol. 36, April 1964, pp. 697-706.

[5] Zinn, B.T., Bell, W.A., Daniel, B.R., and Smith, A.J., Jr., "Experimental Determination of Three-Dimensional Liquid Rocket Nozzle Admittances," *AIAA Journal,* Vol. 11, March 1973, pp. 267-272.

Solution Bounds to Structural Systems

V. R. Murthy* and T. J. McDaniel†

Iowa State University, Ames, Iowa

I. Introduction

THE theory of differential and integral inequalities provides a method of constructing bounds to the solution for a wide range of engineering problems. The texts of Walter,[1] Protter and Weinberger,[2] and Lakshmikantham and Leela[3] summarize the major theorems and contain numerous references. In theory it is possible to construct both upper and lower bounds to the solution for certain classes of partial differential equations and to systems of ordinary differential and integral equations. In practice it is not a simple matter to construct such bounds.

Received May 7, 1975; revision received September 19, 1975. This research was sponsored by the National Science Foundation under Grant GK-40589 and by the Iowa State University Engineering Research Institute.

Index category: Structural Stability Analysis.

*Research Assistant Professor of Aerospace Engineering and Engineering Research Institute.

†Associate Professor of Aerospace Engineering and Engineering Research Institute. Member AIAA.

In the present Note a procedure is sought which directly yields such bounds for an arbitrary system of variable coefficient linear differential equations, such that upper and lower bounds agree to a prescribed number of digits. An efficient means of refining these bounds is then required. To illustrate the procedures which are developed, bounds on the buckling load and the buckling shape of a varying geometry column will be obtained.

The theory of differential and integral inequalities applies primarily to initial value problems. Since the transfer matrix method, which has been applied to a variety of engineering problems,[4-6] converts a two point boundary value problem to an initial value problem, the inequality theorems directly apply to linear structures with varying geometry. The inequality theorems have not been developed sufficiently to allow the construction of bounds to two point nonlinear boundary value problems except for special classes of problems.

It is evident from the literature that few engineering applications of the inequality theorems have been carried out. This is partially due to a lack of direct procedures for constructing such bounds. Also, one may suspect that the extra computational effort involved in producing such bounds does not justify the convenience of having such bounds. Actually, the generation of bounds can lead to computational efficiency, which is obtained by using limited accuracy intermediate computations. The accuracy of the final computation can be judged by the number of places of agreement of the upper and lower bounds of that solution.

Some applications of the theory of inequalities to boundary-layer flow,[7-8] flow of viscous fluid in a narrow slit,[9] and heat conduction,[10] have been accomplished. In general, high precision in the results was not demonstrated. In more recent articles[11,12] quite accurate bounds have been obtained by using differential inequalities. Reference 11 computed upper and lower bounds to the response of a nonlinear single degree-of-freedom system which gave good agreement of the bounds (7 or more digits) over the range of computation. This approach does require considerable computation to obtain the required bounds. Reference 12 avoids this difficulty for a specialized system of differential equations associated with the analysis of beams with varying geometry. Rough analytical bounds were improved by Picard iteration to obtain good agreement (4 or 5 digits) between the upper and lower bounds for the required transfer matrix elements. Both upper and lower bounds on the transfer matrix were computed in less time than required for a standard Runge-Kutta procedure to obtain a numerical transfer matrix. Bounds on the solution state vector were also in agreement to 3 or 4 places. We hope to demonstrate the same computational efficiency in obtaining bounds for a more general system of equations considered following.

II. Application of Differential Inequalities

Consider a variable coefficient linear differential equation of the type

$$d/dx[T(x)] = [A(x)]\,[T(x)] \qquad 0 \le x \le \ell \tag{1}$$

$$\text{with initial conditions} \quad [T(0)] = [I] \tag{2}$$

The desired solution to the previous equation is the transfer matrix $[T(x)]$. $[T(x)]$ exists and is unique if the elements of $[A(x)]$ are piecewise continuous and bounded. The elements of $[A(x)]$ in the present general case contain both positive and negative elements. For example, the $[A(x)]$ matrix for a column allowed to buckle about one axis is given by

$$[A(x)] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & b(x) & 0 \\ 0 & -P & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

where $b(x) = 1/EI(x)$

P is the compressive loading and $EI(x)$ is the varying geometry section property. Let $[A(x)]$ be separated into the difference of 2 matrices $[B(x)] - [C(x)]$ where $[C(x)]$ contains the magnitude of all negative elements of $[A(x)]$ except for negative diagonal elements. The matrix $[B(x)]$ contains all positive terms of $[A(x)]$ and the negative diagonal terms. We then denote $[\hat{T}(x)]$ and $[\tilde{T}(x)]$ as the desired upper and lower bounds to $[T(x)]$. According to Ref. 1, the necessary conditions under which $[\hat{T}(x)] \geq [T(x)] \geq [\tilde{T}(x)]$ holds in the interval $0 \leq x \leq \ell$ are

$$d/dx[\hat{T}(x)] \geq [B(x)]\,[\hat{T}(x)] - [C(x)]\,[\tilde{T}(x)] \tag{4}$$

in $0 \leq x \leq \ell$

$$d/dx[\tilde{T}(x)] \leq [B(x)]\,[\tilde{T}(x)] - [C(x)]\,[\hat{T}(x)] \tag{5}$$

in $0 \leq x \leq \ell$

and

$$[\hat{T}(0)] = [I] = [\tilde{T}(0)] \tag{6}$$

Inequality can also be allowed in Eq. (6) but better bounds are obtained if equality is required. Multiplying Eq. (5) by -1 reverses the sense of that inequality and allows us to write

$$\frac{d}{dx}\begin{bmatrix} \hat{T}(x) \\ -\tilde{T}(x) \end{bmatrix} \geq \begin{bmatrix} B(x) & C(x) \\ C(x) & B(x) \end{bmatrix} \begin{bmatrix} \hat{T}(x) \\ -\tilde{T}(x) \end{bmatrix}$$

$$\text{with} \begin{bmatrix} \hat{T}(0) \\ -\tilde{T}(0) \end{bmatrix} = \begin{bmatrix} I \\ -I \end{bmatrix} \tag{7}$$

To avoid the problem of negative initial conditions we solve

$$\frac{d}{dx}[\hat{R}(x)] \geq \begin{bmatrix} B(x) & C(x) \\ C(x) & B(x) \end{bmatrix} [\hat{R}(x)] \text{ with } [\hat{R}(0)] = [I]$$

$$\geq [D(x)]\,[\hat{R}(x)] \tag{8}$$

Multiplying Eq. (4) by -1 we can also form

$$\frac{d[\tilde{R}(x)]}{dx} \leq [D(x)]\,[\tilde{R}(x)] \text{ with } [\tilde{R}(0)] = [I] \tag{9}$$

The bounds $[\hat{R}(x)]$ and $[\tilde{R}(x)]$ are required in order to obtain $[\hat{T}]$ and $[\tilde{T}]$. Since $[D(x)]$ can have negative elements only on the diagonal it can be shown that $[R(x)]$ contains only positive elements. This can be shown by representing the solution $[R(x)]$ in terms of the matrizant of $[D(x)]$.

The procedures for obtaining a rough bound on $[R(x)]$ is to choose the maximum value of all elements in $[D(x)]$ to form a constant coefficient matrix $[\hat{D}]$. The solution to

$$\frac{d[\hat{R}(x)]}{dx} = [\hat{D}]\,[\hat{R}(x)] \text{ with } [\hat{R}(0)] = [I] \tag{10}$$

provides an equation with constant coefficients that can be readily solved and by inserting Eq. (10) into Eq. (8) satisfies the desired inequality.

A lower bound is obtained by solving

$$\frac{d[\tilde{R}(x)]}{dx} = [\tilde{D}]\,[\tilde{R}(x)] \text{ with } [\tilde{R}(0)] = [I] \tag{11}$$

where $[\tilde{D}]$ contains minimum values of $[D(x)]$ in the given interval. $[\tilde{R}(x)]$ then satisfies Eq. (9).

To improve the bound $[\hat{R}(x)]$ a Picard iteration procedure is used

$$[\hat{R}_1(x)] = \int_0^x [D(\bar{x})]\,[\hat{R}(\bar{x})]d\bar{x} + [I] \tag{12}$$

Due to properties of $[D(x)]$ it is easily shown that $[\hat{R}_1(x)]$ is also an upper bound to $[R(x)]$. By differential inequalities it can be shown that $[\hat{R}_1(x)]$ is a lower bound to $[\hat{R}(x)]$ thus it is a better bound. Further iterations improves the bound to a desired accuracy. The lower bound $[\tilde{R}(x)]$ can be iterated to a desired accuracy by this procedure. Computation efficiency results if the original bound and the subsequent iterations can be obtained in algebraic form and evaluated numerically as a final operation.

The recovery of $[\hat{T}]$ and $[\tilde{T}]$ follows the rules for the matrix multiplication of bounds is known. Suppose upper and lower bounds on matrices $[A]$ and $[B]$ are known where

$$\begin{aligned} [\hat{A}] &= [\hat{A}+] - [\hat{A}-], & [\tilde{A}] &= [\tilde{A}+] - [\tilde{A}-] \\ [\hat{B}] &= [\hat{B}+] - [\hat{B}-], & [\tilde{B}] &= [\tilde{B}+] - [\tilde{B}-] \end{aligned} \tag{13}$$

The terms $[\hat{A}+]$ and $[\tilde{A}-]$ for example contain respectively the magnitudes of the positive and negative elements of $[\hat{A}]$. An upper bound matrix $[C]$ where $[C] = [A]\,[B]$ is given by

$$\begin{aligned} [\hat{C}] &= [\hat{C}+] - [\hat{C}-] \\ [\hat{C}+] &= [\hat{A}+]\,[\hat{B}+] + [\tilde{A}-]\,[\tilde{B}-] \\ [\hat{C}-] &= -[\tilde{A}+]\,[\hat{B}-] - [\hat{A}-]\,[\tilde{B}+] \end{aligned} \tag{14}$$

The lower bound $[\tilde{C}]$ is obtained by interchanging $\hat{}$ and $\tilde{}$ in Eqs. (14). It then follows from Eqs. (7) and (14) and the fact that all elements of $[R]$ are positive that

$$\begin{aligned} [\hat{T}(x)] &= [\hat{R}_{11}(x)] - [\tilde{R}_{12}(x)] \\ [\tilde{T}(x)] &= [\tilde{R}_{11}(x)] - [\hat{R}_{12}(x)] \end{aligned} \tag{15}$$

where $[R]$ has been partitioned into 4 matrices $[R_{11}]$, $[R_{12}]$, $[R_{21}]$, $[R_{22}]$ of half the dimension of $[R]$. We now have the upper and lower bounds for the transfer matrix elements for a general system of linear differential equations. These bounds can be further improved by additional Picard iterations before using Eq. (15). In Sec. III we will apply the bounds on the transfer matrix to the determination of bounds on the buckling load and buckling shape of varying geometry columns.

III. Bounds for Column Buckling

To obtain bounds on the buckling load of a varying geometry column, upper and lower bounds on the transfer matrix for that beam are required. Starting with the $[A(x)]$ matrix in Eq. (3) the $[B(x)]$ and $[C(x)]$ matrices of Eq. (4) are obtained. One can then proceed to Eq. (10) where the maximum value of $b(x)$, i.e. $\hat{b}$, in the given interval is inserted in $[\hat{D}]$. The eigenvalues of $[\hat{D}]$ are easily found to be $\hat{\lambda}_{1,2,3,4} = 0$, $\hat{\lambda}_{5,6} = \pm(p\hat{b})^{1/2}$, $\hat{\lambda}_{7,8} = \pm i(p\hat{b})^{1/2}$. The solution which is upper bound, i.e., $[\hat{R}(x)]$, is then easily obtained. Setting $b(x)$ to its minimum value, i.e. $\tilde{b}$ one can then solve for $[\tilde{R}(x)]$. One procedure for obtaining a better bound on $[R(x)]$ is to divide the beam into N segments and find $\hat{b}_i$ and $\tilde{b}_i$ and the corresponding $[\hat{R}_i(x)]$ and $[\tilde{R}_i(x)]$ for $i=1,2,...N$. According to the rules for obtaining bounds on matrix products the upper and lower bound $[R]$ matrices for the entire beam $[\hat{R}^*]$ and $[\tilde{R}^*]$ are given by

$$[\hat{R}^*_{(\ell)}] = [\hat{R}_N(\ell_N) ... [\hat{R}_2(\ell_2)] [\hat{R}_1(\ell_1)]]$$

$$[\tilde{R}^*_{(\ell)}] = [\tilde{R}_N(\ell_N)] ... [\tilde{R}_2(\ell_2)] [\tilde{R}_1(\ell_1)]^{\ell=\ell_1+\ell_2...+\ell_N} \quad (16)$$

The simple form of Eq. (16) results from the fact that all quantities are positive.

A more efficient procedure for improving the bounds is to Picard iterate the initial bounds one or more times using Eq. (12). This iteration can be carried out for the entire length of beam or in segments of the beam and then use Eqs. (16) if the upper and lower bounds are to agree to several places. Again using Eq. (15) the transfer matrix bounds are obtained.

Application of the transfer matrix method to determine the buckling load of a cantilever column is straightforward. The boundary conditions for this column are $W(0)=\psi(0)=0$ and $M(\ell)=V(\ell)=0$ where $-W, \psi, M, V$ are elements of the usual state vector[4] for the transfer matrix analysis of beam-columns. When one applies the previous boundary conditions, buckling is defined by $t_{33}(\bar{P})=0$ where $\bar{P}$ is the buckling load. Since only bounds on the transfer matrix elements are known, the upper and lower bounds on the first buckling load are given by $\hat{t}_{33}(\hat{P})=0$ and $\tilde{t}_{33}(\tilde{P})=0$.

In Table 1, 3 different means of computing bounds on the first buckling load of a varying geometry column are compared. The parameter $b(x)$ was assumed to vary as $b(x)=b_1x+b_2$ in this and following tables. One can see from the table that dividing the column into 10 segments yields slightly better bounds on the buckling load than Picard iterating a rough bound twice over the entire length of the column. A combination of iteration and segmenting yields close bounds on a double precision Runge-Kutta solution for the buckling load.

Bounds on the buckling load of a cantilever column as a function of length is given in Table 2. It is evident that the closeness of the bounds is approximately constant for the range of lengths investigated. Bounds on the buckling load agree to 3 digits whereas the transfer matrix bounds from which they were obtained agreed in general to 4 places.

In Table 3 upper and lower bounds on the buckled shape of the column discussed in Table 1 are presented. A unit deflection is assumed at the free end of the cantilever column for the unknown exact solution. Expressions for bounds on the deflected shape are obtained by using the rules for the division of bounds such that

$$\hat{W}(x)=\hat{t}_{13}(x)/\tilde{t}_{13}(\ell)$$
$$\tilde{W}(x)=\tilde{t}_{13}(x)/\hat{t}_{13}(\ell) \quad (17)$$

since $t_{13}(x)$ is positive in $0 \le x \le \ell$.

IV. Discussion

A direct procedure for the construction of both upper and lower bounds to the solution of an arbitrary system of variable coefficient differential equations has been obtained. Picard iteration is used to further improve both upper and lower bounds independently. As an illustration, bounds on the buckling load and buckled shape of varying geometry cantilever column were obtained. A linear variation of $b(x)$ was used in this illustration since many types of $b(x)$ can be approximated by linear variations in a given segment of the column. If this linear approximation is an upper bound to the actual $b(x)$ then an upper bound to $[R(x)]$ is assured in this segment. A corresponding lower bound to $[R(x)]$ is obtained from a linear approximation to $b(x)$ which is its lower bound. The efficiency of the present approach depends upon the Picard iteration to obtain close bounds without excessive computations. The advantage of the present approach is that limited accuracy intermediate computations can be used with confidence since upper and lower bounds on the final result are obtained.

Table 1 Buckling loads $\ell=50.0$ in., $b_1=-10^{-6}/3$ (lb-in.^{3})$^{-1}$, $b_2=10^{-4}/3$ (lb-in.^{2})$^{-1}$

10 segments, no iterations		1 segment, 2 iterations		10 segments, 2 iterations		Runge-Kutta solution
Ub.	Lb	Ub.	Lb.	Ub.	Lb.	Lb.
37.290	32.764	36.450	31.200	34.733	34.717	34.725

Table 2 Buckling loads $b_1=10^{-6}/3$ (lb-in.^{3})$^{-1}$, $b_2=10^{-4}/3$ (lb-in.^{2})$^{-1}$

	10 Sections—2 Iterations		
Length in.	Upper bound Lb.	Runge-Kutta solution Lb.	Lower bound Lb.
30.0	90.264	90.255	90.240
40.0	52.488	52.481	52.469
50.0	34.733	34.725	34.717
60.0	24.966	24.960	24.952
70.0	19.048	19.044	19.036

Table 3 Mode shape $\ell=50.0$ in., $a=10^{-6}/3$ (lb-in.^{3})$^{-1}$, $c=10^{-4}/3$ (lb-in^2)$^{-1}$

Station in.	Upper bound	Runge-Kutta solution	Lower bound
0.0	0.000000	0.000000	0.000000
10.0	0.055444	0.055425	0.055406
20.0	0.208668	0.208590	0.208515
30.0	0.435008	0.434833	0.434662
40.0	0.707344	0.707033	0.706725
50.0	1.000490	1.000000	0.999510

References

[1] Walter, W., *Differential and Integral Inequalities*, Springer Verlag, N. Y., 1970.

[2] Protter, M. H. and Weinberger, H. S., *Maximum Principles in Differential Equations*, Prentice Hall, N. J., 1967.

[3] Lakshmikantham, W. and Leela, S., *Differential and Integral Inequalities, Vol. I, Ordinary Differential Equations*, Academic Press, N. Y. and London, 1969.

[4] Pestel, E. C. and Lecki, F. A., *Matrix Methods in Elastomechanics*, McGraw-Hill, N. Y., 1963.

[5] Lin, Y. K., *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, N. Y., 1963.

[6] McDaniel, T. J. and Henderson, J. P., "Review of Transfer Matrix Vibration Analysis of Skin Stringer Structure," *The Shock and Vibration Digest*, Vol. 6-1 (Part I), Vol. 6-2 (Part II), 1974.

[7] Nickel, K., "Eine Einfache Abschatzung fur Grezschichten," *Ingenieur Archiv*, Vol. 31, 1962, pp. 85-100.

[8] Adams, E., "Rigorous Bounds of Exact Solutions of Boundary Layer Problems of Prandtl Type," *Acta Mechanica*, Vol. 3, 1967, pp. 268-277.

[9] Wheeler, L., "Flow Rate Estimates for Rectilinear Pipe Flow," *Journal of Applied Mechanics*, Vol. 41, Dec. 1974, pp. 903-906.

[10] Canon, J. R., "Numerical Solutions of Non-Linear Differential Equations," Proceedings of an Advanced Symposium conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin, Madison, Wis., 1966, pp. 21-54.

[11] Bell, C. A. and Appl, F. C., "Bounds for Initial Value Problems," *Journal of Applied Mechanics*, Vol. 40, 1974, pp. 1097-1102.

[12] McDaniel, T. J. and Murthy, V. R., "Solution Bounds for Varying Geometry Beams," *Journal of Sound and Vibration* (to appear), 1976.